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OPTIMAL 3-DIMENSIONAL MINIMUM TIME TURNS FOR AN AIRCRAFT

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13. ABSTRACT

Using a 3-dimensional formulation for an aircraft's dynamics, the required controls for a minimum time-to-turn are calculated. Three controls are used: (1) angle of attack, (2) bank angle, and (3) thrust. The minimum time-to-turn solutions are subject to varying terminal conditions on both flight path angle and heading angle. In general, the times for the turns are not greatly changed by varying thrust/weight ratios or the final flight path angle. Significant effects on the change in total energy, final altitude, final velocity and control histories are noted for variations of the above parameters. Solutions to the above are accomplished through the use of Miele's Sequential Gradient-Restoration Algorithm.

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Optimal 3-Dimensional Minimum

Time Turns for an Aircraft[1]

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Abstract. Using a 3-dimensional formulation for an aircraft's dynamics, the required controls for a minimum time-to-turn are calculated. Three controls are used: (1) angle of attack, (2) bank angle, and (3) thrust. The minimum time-to-turn solutions are subject to varying terminal conditions on both flight path angle and heading angle.

In general, the times for the turns are not greatly changed by varying thrust/weight ratios or the final flight path angle. Significant effects on the change in total energy, final altitude, final velocity and control histories are noted

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for variations of the above parameters.

Solutions to the above problem are accomplished through the use of Miele's Sequential Gradient-Restoration Algorithm.

1. Introduction

In the past few years, optimization techniques have been used to determine turning performance of high speed aircraft. Refs. 1-3 discuss minimum-fuel and minimum-time turns at constant altitude. Bryson and Hedrick (Refs. 4-5) have extended the analysis of these types of turns to three dimensions by using the energy-state approximation. The energy-state approximation was also used by Beebee (Ref. 6) to determine three-dimensional, minimum-time turns for a hypersonic rocket powered aircraft. Kelly and Edelbaum (Ref. 7) suggested the use of asymptotic expansions in conjuction with energy-climbs and energy-turns.

Energy approximations are very helpful in studying optimal aircraft flight since the order of the state differential system is reduced. In the case of minimum-time-to-climb, Bryson, Desai, & Hoffman (Ref.8) have shown that energy-approximations give good

results when compared to the more complex dynamic models. Yet, there is a need to check the accuracy of these approximations when used to determine three-dimensional turning performance.

Preyss, Willes, Humphreys, and Roberts (Ref. 9) applied the Pontryagin Minimum Principle to the problem. This approach to minimum time turns results in the usual two-point boundary value problem with its inherent solution difficulties. The approach used in this paper is based on the Sequential Gradient-Restoration Algorithm developed by Miele , Pritchard and Damoulakis in Reference 10.

2. List of Symbols

x_1	non-dimensional distance (x-direction)
x_2	non-dimensional distance (y-direction)
x_3	non-dimensional distance (z-direction)
x_4	non-dimensional speed
x_5	flight-path angle (γ)
x_6	heading angle
u_1	pseudo angle-of-attack controller
u_2	bank angle controller
u_3	pseudo thrust controller
L	non-dimensional lift
D	non-dimensional drag

σ	non-dimensional density function
k_1	normalized profile drag coefficient
k_2	normalized induced drag coefficient
k_3	air density height coefficient
W	normalized weight
T	normalized thrust
α	normalized angle-of-attack
λ_i	adjoint associated with state x_i
θ	independent time variable
t	independent time variable for fixed end time formulation
L_{x_i}	$\frac{\partial L}{\partial x_i}$
L_{u_1}	$\frac{\partial L}{\partial u_1}$
D_{x_i}	$\frac{\partial D}{\partial x_i}$
D_{u_1}	$\frac{\partial D}{\partial u_1}$
n	normalized load limit
v_c	corner velocity

3. Formulation of the Problem

The objective of this paper is to determine the optimal control required for an aircraft to make a minimum time-to-turn in three dimensions. In optimal control notation, one desires to find the minimum of the cost functional

$$I = \theta_f \tag{1}$$

subject to the differential constraints

$$\frac{dx}{d\theta} = \varphi(x,u,\theta) \tag{2}$$

and subject to the boundary conditions

$$\begin{aligned} x(0) - x_o &= 0 \\ \Psi(x,\theta)\Big|_{\theta_f} &= 0 \end{aligned} \tag{3}$$

where the state variable x is an n-vector, the control variable u is an m-vector, θ is the independent time variable, θ_f is the final time and Ψ is a q-vector function defining the terminal boundary.

3.1 Equivalent Fixed Final Time Problem The Sequential Gradient-Restoration Algorithm was formulated for the final time fixed. This transformation is always possible through the use of a parameter (see Ref. 10). In this problem, if the time θ is redefined by the transformation

$$\theta = \tau t \tag{4}$$

where $\tau=\theta_f$, then equations 1-3 may be written as[5]

$$I = \tau \tag{5}$$

---

[5]Differentiation by the independent variable t is denoted by a dot (.).

$$\dot{x} = \varphi(x, u, \tau, t) \tag{6}$$

$$x(0) - x_0 = 0 \tag{7}$$

$$\psi(x, \tau)_1 = 0 \tag{8}$$

3.2 <u>Aircraft Equations of Motion</u> If one assumes (a) that the aircraft flies a coordinated turn, (b) that negligible fuel is consumed during the maneuver, and (c) that the error due to the thrust vector not being colinear with the velocity vector is negligible, then the equations of motion in the velocity axis coordinate system are

$$\begin{aligned}
\dot{x}_1 &= \tau x_4 \cos(x_5) \cos(x_6) \\
\dot{x}_2 &= \tau x_4 \cos(x_5) \sin(x_6) \\
\dot{x}_3 &= -\tau x_4 \sin(x_5) \\
\dot{x}_4 &= \tau(T - D - W\sin(x_5)) \\
\dot{x}_5 &= \tau(L\cos(u_2) - W\cos(x_5))/x_4 \\
\dot{x}_6 &= \tau L\sin(u_2)/(x_4\cos(x_5))
\end{aligned} \tag{9}$$

The initial conditions of the state variable x are assumed given. The vector function ψ for the problem to be studied is given by

$$\psi(x,\tau) = \begin{Bmatrix} x_5(1) - x_{f5} \\ x_6(1) - x_{f6} \end{Bmatrix} = 0 \tag{10}$$

where the components x_1 through x_4 are free at the final time. The equations defining L, D, and σ are

$$\begin{aligned} L &= \alpha\sigma x_4^2 \\ D &= (k_1+k_2\alpha^2)\,\sigma\, x_4^2 \\ \sigma &= e^{k_3 x_3} \end{aligned} \tag{11}$$

3.3 Control Constraints In order to obtain realistic results, it is necessary to keep the angle-of-attack, the thrust, and the "g" loading constrained within specified limits. These constraints are specified in this problem by

$$\begin{aligned} 0 &\le \alpha \le 1 \\ 0 &\le \alpha\sigma x_4^2 \le 1 \\ 0 &\le T \le T_{max} \end{aligned} \tag{12}$$

All three of these constraints result in control variable inequality constraints and may be handled by a simple substitution transformation. For equation 12(c) the transformation is

$$T = T_{max}\sin^2(u_3) \tag{13}$$

Equations 12(a) and 12(b) both result in a constraint on the angle-of-attack . These two constraints can be seen pictorially in Figure 1.

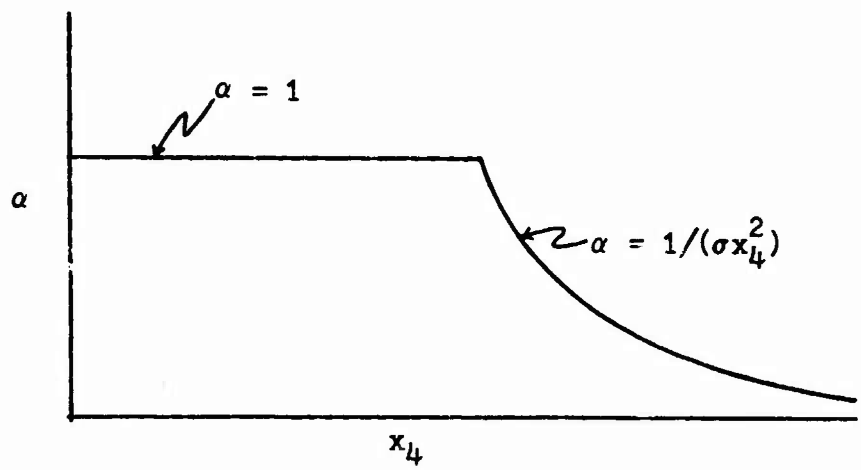

Figure 1. Angle-of-Attack vs Velocity Constraint

For velocities below the corner velocity, $\sigma^{-\frac{1}{2}}$, the transformation

$$\alpha = \sin^2(u_1) \tag{14}$$

is used, and for velocities above the corner velocity the transformation

$$L = \sin^2(u_1) \tag{15}$$

is used. The same control u_1 is used for both inequality conditions, equations 12(a) and 12(b), since these two conditions do not hold simultaneously.

4. Solution Procedure

The solution procedure consists of alternate applications of a gradient phase and restoration phases. In the gradient phase, nominal functions for x(t), u(t), and τ which satisfy the differential constraints and boundary conditions are assumed. By minimizing the first-order change to the performance index, variations $\Delta x(t)$, $\Delta u(t)$, and $\Delta\tau$ are determined. These variations are added to the nominal functions to obtain varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, and $\tilde{\tau}$ which produce a decrease in the performance index. Since the constraints are satisfied only to first order in this phase, the varied functions normally do not satisfy the non-linear differential equations or the boundary conditions. Consequently, a restoration phase is used to change $\tilde{x}(t)$, $\tilde{u}(t)$, and $\tilde{\tau}$ so that the non-linear differential equations and boundary conditions are satisfied.

To begin the restoration phase, the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, and $\tilde{\tau}$ are treated as nominal functions. By minimizing the least-square change of $\tilde{u}(t)$ and $\tilde{\tau}$ subject to the quasi-linearized differential equations and quasilinearized boundary

conditions, a new set of variations $\Delta\tilde{x}(t)$, $\Delta\tilde{u}(t)$, and $\Delta\tilde{\tau}$ is obtained. These variations are added to the nominal functions $\tilde{x}(t)$, $\tilde{u}(t)$, and $\tilde{\tau}$ to obtain new varied functions $\hat{x}(t)$, $\hat{u}(t)$, and $\hat{\tau}$. This step is performed iteratively until the differential equations and boundary conditions are restored to some preselected degree of accuracy. The cumulative errors in the constraints and optimum conditions are evaluated using the functionals[6]

$$P = \int_0^1 N(\dot{x} - \varphi)dt + N(\Psi_1) \tag{16}$$

$$Q = \int_0^1 N(\dot{\lambda} + \varphi_x\lambda)dt + \int_0^1 N(f_u - \varphi_u\lambda)dt + N\left(\int_0^1 (f_\tau - \varphi_\tau)dt + (1+\psi_\tau\nu)_1\right) + N(\lambda+\psi_x\nu)_1 \tag{17}$$

Thus P must be reduced to an acceptably small value in each cycle of the Sequential Gradient-Restoration Algorithm. When both P and Q are acceptably small values, convergence has been obtained.

[6]The norm of a vector z is defined to be $N(z) = z^T z$.

Initially, nominal solutions which satisfy all the constraint conditions, equations 6 - 8, are not normally known. However, this presents no problem, since only the restoration phase need be applied to some initial guess. In this way, feasible nominal solutions can be generated and the Sequential Gradient-Restoration Algorithm can be used to seek the optimal solution.

5. Necessary Conditions

As in other approaches to optimizaion problems, this method also has necessary conditions which must be satisfied. Both the gradient phase and the restoration phase have a set of necessary conditions which result from application of optimization criteria such as the Euler equations and transversality conditions. However, the particular form of these conditions is a consequence of the numerical technique to be used. A

detailed development of the results presented in the next few paragraphs can be found in Refs. 10 - 11. The necessary conditions are as follows:

5.1 Gradient Phase

$$\dot{A} = \varphi_x^T A + \varphi_u^T B + \varphi_\tau^T C \tag{18}$$

$$\dot{\lambda} = - \varphi_x \lambda \tag{19}$$

$$B = - \varphi_u \lambda \tag{20}$$

$$C = - \int_0^1 \varphi_\tau \lambda dt - 1 \tag{21}$$

where $A=\Delta x/a$, $B=\Delta u/a$, $C=\Delta\tau/a$, and a is the step size for the gradient phase. Equation (18) is a first-order approximation of the differential constraints and equations (19) - (21) are the necessary conditions. The boundary conditions are

$$A(0) = 0 \tag{22}$$

$$(\psi_x^T A + \psi_\tau^T C)_1 = 0 \tag{23}$$

The equivalence to the transversality condition is

$$(\lambda + \Psi_x \nu)_1 = 0 \tag{24}$$

5.2 Restoration Phase

$$\dot{\lambda} = \varphi_x^T A + \varphi_u^T B + \varphi_\tau^T C - (\varphi - \dot{x}) \tag{25}$$

$$\dot{\tilde{\lambda}} = -\varphi_x \tilde{\lambda} \tag{26}$$

$$B = -\varphi_u \tilde{\lambda} \tag{27}$$

$$C = -\int_0^1 \varphi_\tau^T \tilde{\lambda} dt \tag{28}$$

where $A=\Delta\tilde{x}/b$, $B=\Delta\tilde{u}/b$, $C=\Delta\tilde{\tau}/b$, and b is the stepsize for the restoration phase. Equation (25) is a quasilinear approximation of the differential constraints and equations (26) - (28) are the necessary conditions. The boundary conditions are

$$A(0) = 0 \tag{29}$$

$$(\Psi + \Psi_x^T A + \Psi_\tau^T C)_1 = 0 \tag{30}$$

The equivalence to the transversality condition is

$$(\tilde{\lambda} + \Psi_x \nu)_1 = 0 \tag{31}$$

The matrices φ_x^T, φ_u^T, and φ_τ^T are given by the same expression in both the gradient and restoration phases. For the matrix φ_x^T, all the elements $\varphi_x^T(ij)$ are zero except for the following:

$$
\begin{aligned}
(1,4) &= \tau\cos(x_5)\cos(x_6)\\
(1.5) &= -\tau x_4\sin(x_5)\cos(x_6)\\
(1,6) &= -\tau x_4\cos(x_5)\sin(x_6)\\
(2,4) &= \cos(x_5)\sin(x_6)\\
(2,5) &= -\tau x_4\sin(x_5)\sin(x_6)\\
(2,6) &= \tau x_4\cos(x_5)\cos(x_6)\\
(3,4) &= -\tau\sin(x_5)\\
(3,5) &= -\tau x_4\cos(x_5)\\
(4,3) &= -\tau D_{x_3}\\
(4,4) &= -\tau D_{x_4}\\
(4,5) &= -\tau W\cos(x_5) \qquad (32)\\
(5,3) &= \tau L_{x_3}\cos(u_2)/x_4\\
(5,4) &= -\tau(L\cos(u_2)-W\cos(x_5))/x_4^2 + \tau L_{x_4}\cos(u_2)/x_4\\
(5,5) &= \tau W\sin(x_5)/x_4\\
(6,3) &= \tau L_{x_3}\sin(u_2)/x_4\cos(x_5)\\
(6,4) &= -\tau L\sin(u_2)/x_4^2\cos(x_5) + \tau L_{x_4}\sin(u_2)/x_4\cos(x_5)\\
(6,5) &= \tau L\sin(u_2)\sin(x_5)/x_4\cos^2(x_5)
\end{aligned}
$$

For the matrix φ_u^T, all the elements are zero except for the following:

$$
\begin{aligned}
(4,1) &= -\tau D_{u_1}\\
(4,3) &= 2\tau T_{max}\sin(u_3)\cos(u_3)
\end{aligned}
$$

$$(5,1) = \tau\cos(u_2)L_{u_1}/x_4$$

$$(5,2) = -\tau L\sin(u_2)/x_4 \tag{33}$$

$$(6,1) = \tau\sin(u_2)L_{u_1}/x_4\cos(x_5)$$

$$(6,2) = \tau L\cos(u_2)/x_4\cos(x_5)$$

For the matrix φ_τ^T, the elements are:

$$\begin{aligned}
(1,1) &= x_4\cos(x_5)\cos(x_6) \\
(2,1) &= x_4\cos(x_5)\sin(x_6) \\
(3,1) &= -x_4\sin(x_5) \\
(4,1) &= T-D-W\sin(x_5) \\
(5,1) &= (L\cos(u_2)-W\cos(x_5))/x_4 \\
(6,1) &= L\sin(u_2)/(x_4\cos(x_5))
\end{aligned} \tag{34}$$

where, for $x_4 \leq \sigma^{-\frac{1}{2}}$,

$$L_{x_3} = k_3 L \; ; \quad L_{x_4} = 2L/x_4 \; ; \quad D_{x_3} = k_3 D \; ; \quad D_{x_4} = 2D/x_4$$

$$L_{u_1} = k_2\, x_4^2\sin(2u_1) \; ; \quad D_{u_1} = 2k_2 L L_{u_1}/\, x_4^2 \tag{35}$$

and for $x_4 \geq \sigma^{-\frac{1}{2}}$,

$$L_{x_3} = 0 \; ; \quad L_{x_4} = 0 \; ; \quad D_{x_3} = k_3(k_1\, x_4^2-k_2L^2/\, x_4^2)$$

$$D_{x_4} = 2(k_1\, x_4^2-k_2L^2/\, x_4^2)/x_4 \; ; \quad L_{u_1} = \sin(2u_1) \tag{36}$$

$$D_{u_1} = 2k_2 L L_{u_1}/\, x_4^2$$

6. Numerical Results

The numerical results presented are for an airplane with the following nominal characteristics:

$W = 12{,}150$ lbs.	$k_1 = 0.02$
$n_{max} = 7.22$	$k_2 = 0.05$
$S = 237$ sq.ft.	$C_{L_{max}} = 1.0$

The aircraft was assumed to be flying straight and horizontal immediately before entering the turn. Thus, the flight path angle and heading angle were given initial values of zero. The position coordinates x_1 and x_2 were assigned values of zero and the altitude was specified as 13,390 ft[7].

In order to obtain information on a variety of turning situations, the thrust/weight ratio, initial velocity, final heading, and final flight path angle were all varied as shown in Table 1. Emphasis was placed on a thrust/weight ratio of one since advanced aircraft are expected to have thrust/weight ratios in this range. The initial velocity was varied to give values approximately 130 fps above and below the corner velocity

[7] Note that in the system of equations, altitude is negative upward. Therefore, the initial altitude in the computer simulation was equivalent to -13,390 ft.

Data Set	T/W	$x_4(0)$	$x_6(\tau)$	$x_5(\tau)$
1	0.38	621	180	0
2	0.50	↓	↓	↓
3	0.75	↓	↓	↓
4	1.00	↓	↓	↓
5	1.25	↓	↓	↓
6	1.50	↓	↓	↓
7	0.38	903	180	0
8	0.50	↓	↓	↓
9	0.75	↓	↓	↓
10	1.00	↓	↓	↓
11	1.25	↓	↓	↓
12	1.50	↓	↓	↓
13	1.00	621	180	30
14	↓	↓	↓	15
15	↓	↓	↓	0
16	↓	↓	↓	-15
17	↓	↓	↓	-30
18	1.00	903	180	30
19	↓	↓	↓	15
20	↓	↓	↓	0
21	↓	↓	↓	-15
22	↓	↓	↓	-30
23	1.00	621	90	30
24	↓	↓	↓	15
25	↓	↓	↓	0
26	↓	↓	↓	-15
27	↓	↓	↓	-30
28	1.00	903	90	30
29	↓	↓	↓	15
30	↓	↓	↓	0
31	↓	↓	↓	-15
32	↓	↓	↓	-30

Table 1. Computer Schedule

at the initial altitude. An envelope of 60° for the final flight path angle was obtained by varying this final condition from -30° to $+30^{\circ}$.

In addition to the variables mentioned in the problem formulation, two others which are important in combat maneuvers were evaluated in the numerical process. The change in specific energy was calculated using

$$\Delta h_e = h_e(\tau) - h_e(0) \tag{37}$$

where $h_e = -x_3 + x_4^2/2g$. Also, a turning radius for the 180° turns was calculated using

$$r = \tfrac{1}{2}\left[(x_1(\tau)-x_1(0))^2+(x_2(\tau)-x_2(0))^2+(x_3(\tau)-x_3(0))^2\right]^{\frac{1}{2}} \tag{38}$$

6.1 <u>Effect of Thrust/Weight Ratio</u> To determine the effect of the thrust/weight ratio, the final flight path angle and final heading angle were given values of 0° and 180° respectively. Two cases were considered: (1) initial velocity below the corner velocity and (2) initial velocity above the corner velocity. In both cases, the thrust/weight ratio was varied from 0.38 to 1.5. These two cases are represented by data sets 1 through 12 in Table 1. The corresponding values for the

Data Set	Alt_f(ft)	Vel_f(fts)	e_f(sec)	h_e(ft)	r'ft)
1	9,982	757	10.7	-496	2356
2	10,452	775	10.8	394	2397
3	11,856	796	10.9	2310	2464
4	13,099	828	11.0	4359	2543
5	14,470	859	11.1	6529	2562
6	12,300	794	10.5	2720	2445
7	15,460	784	11.4	-1029	3053
8	16,212	801	11.5	139	3125
9	15,871	809	11.1	-3	2890
10	16,158	828	11.1	769	2853
11	16,856	855	11.1	2577	3058
12	17,634	886	11.2	3798	2883
13	14,499	768	11.1	4275	2612
14	14,146	782	11.0	4273	2562
15	13,099	828	11.0	4359	2542
16	8,829	791	9.5	-780	2296
17	9,041	805	8.5	-295	2211
18	18,970	773	9.9	2196	3003
19	18,940	788	10.9	2534	3034
20	17,250	855	11.1	2577	2853
21	13,303	811	10.9	-2515	2779
22	6,510	1035	10.2	-2910	3495
23	14,491	695	6.4	2603	
24	13,792	719	6.0	2434	
25	13,119	741	5.7	2275	
26	12,571	759	5.5	2144	
27	12,107	776	5.3	2063	
28	15,451	945	6.5	3264	
29	14,588	973	6.5	3274	
30	13,600	1006	6.4	3280	
31	12,519	1040	6.3	3277	
32	11,462	1072	6.2	3264	

Table 2. Computer Results

change in specific energy, turn radius, and the final values for altitude, velocity and time are given in Table 2. As can be seen from Table 2, the thrust/weight ratio has little effect upon the time to turn. However, it does affect final velocity, final altitude, change in specific energy, and turn radius.

When entering the turn with velocity less than the corner velocity (data sets 1 - 6), an increase in thrust/weight ratio produces an increase in final altitude, final velocity, and specific energy. However, the results for a thrust/weight ratio of 1.5 are not consistent with these general trends. This variation from the trend is a result of throttling. Until the thrust/weight ratio reaches 1.5, turns are made a $T=T_{max}$. For a thrust/weight ratio of 1.5, sufficient acceleration exists for the aircraft to fly to the corner velocity, reduce the thrust to zero to remain at the corner velocity, and finally re-apply full thrust to accelerate on the n_{max} boundary to complete the maneuver. The thrust programs for this case are shown in Figure 2.

When entering the turn at a velocity greater than the corner velocity (data sets 7 - 12), an increase in thrust/weight ratio produces an increase in final altitude, final velocity and, in most instances, specific energy. The variations from this trend are also caused by throttling. As shown in Figure 3, throttling occurs when the the thrust/weight ratio is 0.75 or greater.

In the two cases considered, the aircraft must be flown along the stall and/or the n_{max} boundary. Velocity changes are made in the early part of the turn to drive the aircraft velocity toward the corner velocity. If the turn is entered at a velocity less than v_c, the aircraft is flown along the stall boundary until the corner is reached. The remainder of the turn is made on the n_{max} boundary. When entering a turn at velocity greater than v_c, the aircraft is flown along the n_{max} boundary until the corner velocity is reached. The aircraft remains on or about the corner velocity for a short time and then completes the maneuver by accelerating on the n_{max} boundary. In every case, maximum throttle is

applied prior to reaching the corner velocity. As the thrust/weight ratio increases, the maximum throttle is initiated when closer to the corner velocity. Whenever throttling occurs, the aircraft can remain at the corner velocity for a longer period, and consequently, the time to turn is reduced.

The necessary bank angles for these two cases are shown in Figures 4 and 5. As can be seen from these figures, minimum time turns have a given bank angle to start the turn and a continuously varying bank angle throughout the turn. Also, the bank angle is not zero at the end of the turn. Since bank angle is a control variable, it can, by assumption, attain any value instantly.

6.2 <u>Effect of Final Flight Path Angle</u> To determine the effect of the final flight path angle, the thrust/weight ratio was given a value of 1.0. Both 180° and 90° turns with entry speeds above and below v_c were considered. These cases are represented by data sets 13 - 32 in Tables 1 and 2.

6.2.1 <u>180° Turns</u> As shown in Table 2 (data sets 13 - 22), the final flight path angle generally has little effect upon the

time to turn 180°. However, final altitude, final velocity, turn radius, and change in specific energy are strongly affected. When entering the turn with a velocity less than v_c, throttling occurs for the negative final flight path angles (Figure 6). In all cases, however, the turn is started at $T=T_{max}$. When entering the turn with velocity greater than v_c, throttling also occurs. In this case, the turn is started with $T=0$ and power is added after a period of time (Figure 7).

Specifying the final flight path angle has a definite effect upon the type of maneuver required. For example, if the aircraft is flying at a velocity less than v_c, and the final flight path angle is required to be positive, the aircraft will start into a turning dive to gain velocity and end in a climbing turn. An opposite maneuver is required when entering the turn at a velocity greater than v_c and when a negative final flight path angle is specified. This would require a climbing turn through the first part of the turn and a turning dive at the end. The bank angles required for these maneuvers are shown in Figures 8 and 9.

6.2.2 90° Turns For 90° turns, the time to turn decreases as the final flight path angle decreases. This trend holds true for initial velocities above and below the corner velocity (data sets 23 - 32). As the final flight path angle decreases, the aircraft takes advantage of the acceleration of gravity. Thus, in the shorter duration turns, load factor is more important than aircraft velocity. There is no exchange of altitude for velocity to achieve an aircraft velocity near v_c. Instead, the aircraft is flown on the n_{max} boundary with the bank angles shown in Figures 10 and 11.

6.3 Comment on the Sensitivity of Control Variables Analysis of the results of the digital simulation shows that changes in the control variables may have relatively little effect on the minimum time to turn. Although these characteristics are desirable for a pilot performing such maneuvers, they provide minor numerical difficulties in obtaining a solution.

The computer simulations show that the minimum time to turn is not very sensitive to the bank angle control. Relatively large changes in the bank angle program (on the order of several

degrees between itterations) caused virtually no change in the time turn. Precise control of the aircraft in bank angle, therefore, is not required to achieve good results.

The throttling of the thrust is shown as a time varying function over rather short time periods (on the order of a few seconds). As the thrust/weight ratio gets large (greater than or equal to one), the thrust program approaches a bang-bang solution. For intermediate values of thrust/weight ratio (like 0.75), the thrust must be applied well before the corner velocity is attained due to the lag from relatively low acceleration. Higher thrust/weight ratios, therefore, provide for more easily applied thrust programs which is true of bang-bang solutions.

As shown earlier, the angle-of-attack is always limited by the maximum load limit or stall angle-of-attack. These two boundaries are natural ones to a pilot and it is not difficult for a pilot to fly along these boundaries.

6.4 <u>Comments on the Algorithm</u> A characteristic of the numerical algorithm is the ease with which to choose the initial nominal trajectory. For the examples used

in this paper, three constant values of the control are assumed and used to numerically integrate the non-linear equations of motion. After the initial restoration was completed, each intermediate sub-optimal solution always met all the constraints on the problem. Therefore, all solutions are feasible and can be compared directly.

Because of the insensitivity of the optimal solution to the controls, a very strict convergence criterion of $Q \leq 10^{-4}$ was not used. Solutions for which $Q \leq 10^{-2}$ displayed very small changes in the cost functional $I=\tau$.

As a final check on the correctness of the optimal solution, the control histories derived by the Sequential Gradient-Restoration Algorithm were used to numerically integrate the non-linear differential equations. In every case, the solutions obtained from this final integration process matched the optimum solution derived from the Sequential Gradient-Restoration Algorithm.

7. Summary

A method which utilizes the generalized equations of motion to determine three-dimensional minimum-time turns for aircraft

has been presented. The problem was formulated for specified final heading and flight path angles. The analysis was restricted to subsonic flight only by the expression used for drag. The solution procedure and computational procedure were based upon the methods developed in Reference 10.

The system of equations for the problem were programmed in ALGOL for the Burroughs 5500 computer. The computer simulation was used to obtain information on a variety of turning situations. Initial velocity, thrust/weight ratio, final heading, and final flight path angle were found to have a strong influence on the required maneuver. Minimum time turns are always made on the stall and/or n_{max} boundaries. During the turn, throttling from $T=T_{max}$ to $T=0$ or the reverse is frequently required for thrust/weight ratios ≥ 0.75. For low thrust/weight ratios an altitude-velocity exchange is used to approach the corner velocity.

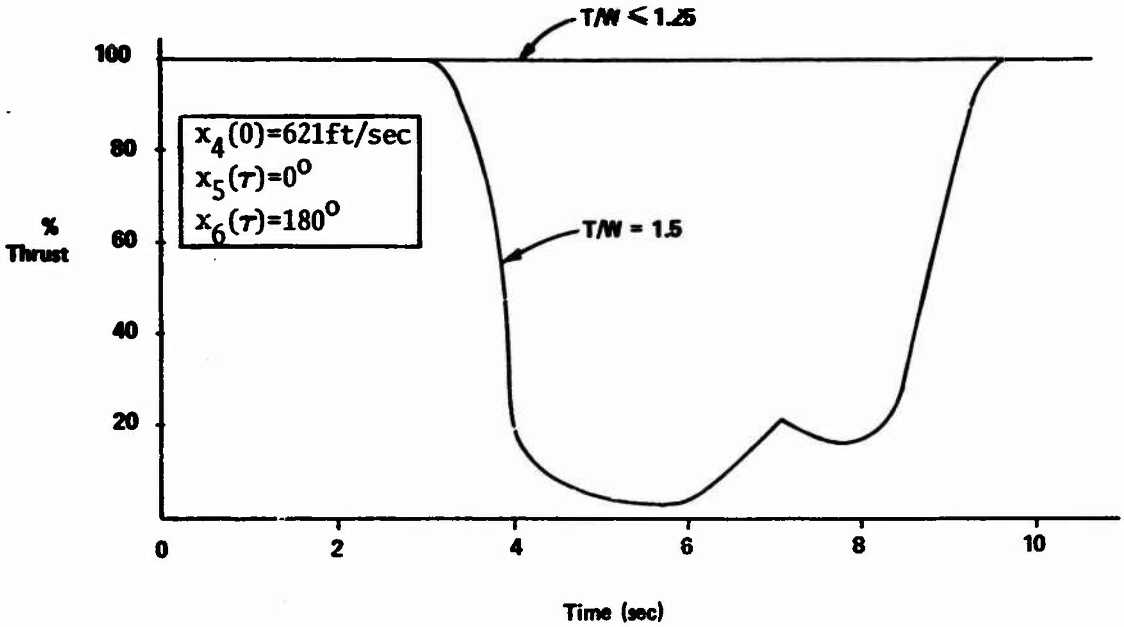

Figure 2. Percent Thrust for Various T/W

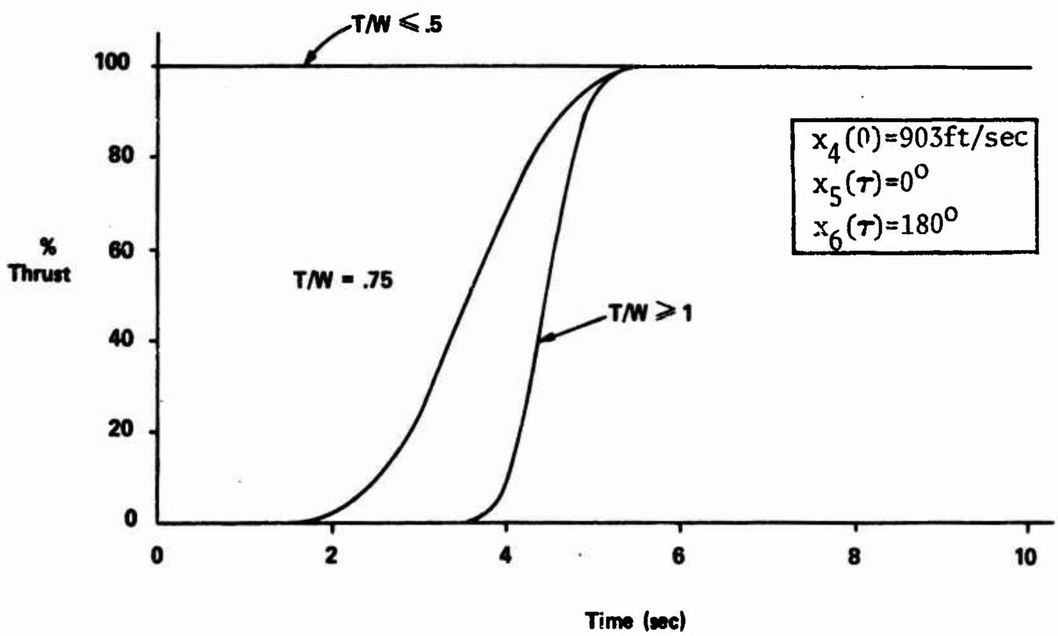

Figure 3. Percent Thrust for Various T/W

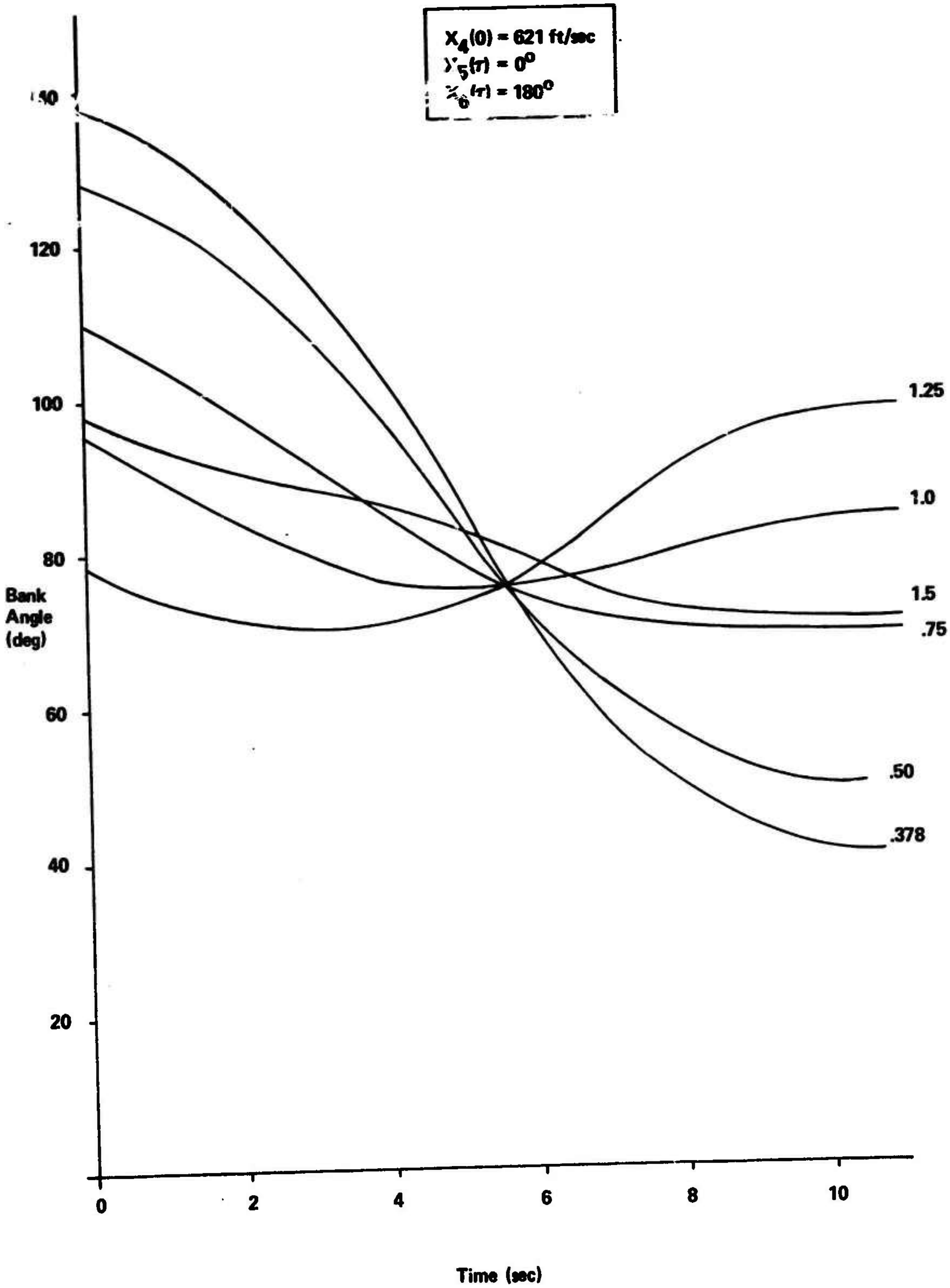

Figure 4. Bank Angle for Various T/W

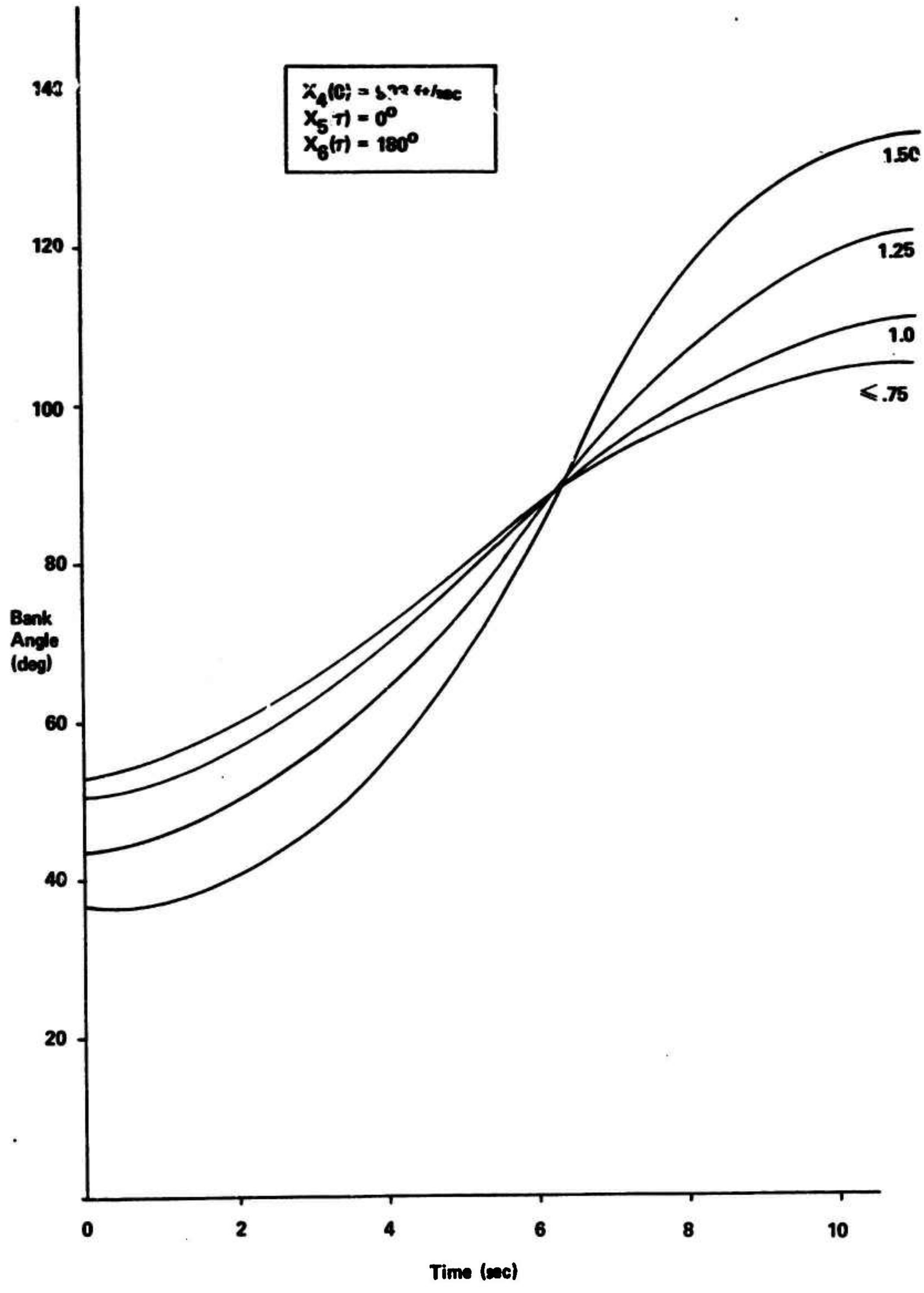

Figure 5. Bank Angle for Various T/W

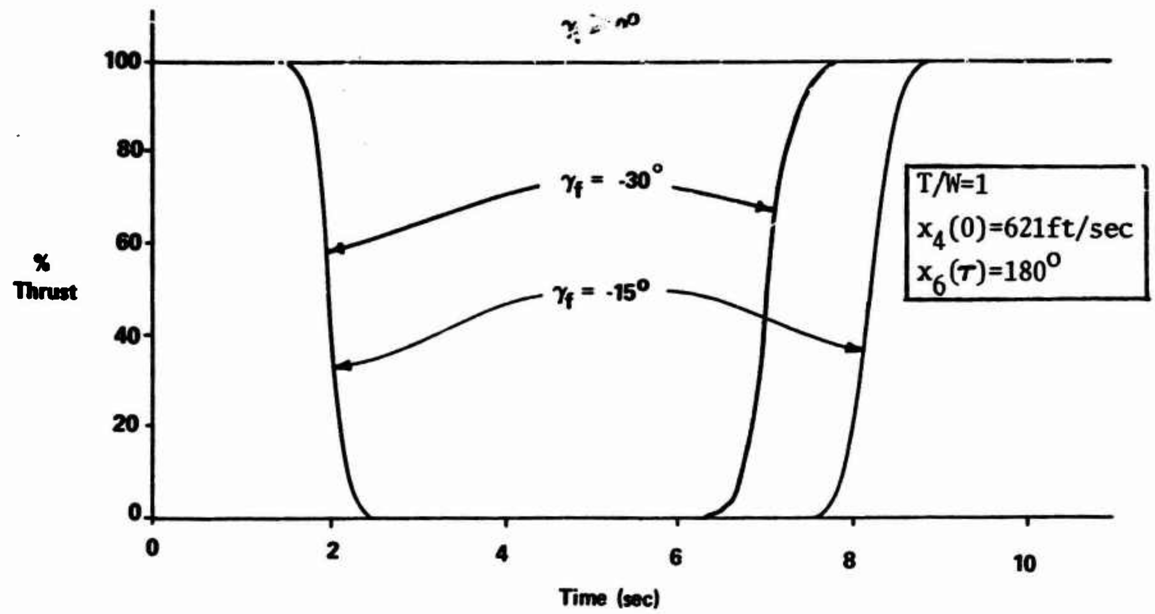

Figure 6. Percent Thrust for Various Final Flight Path Angles

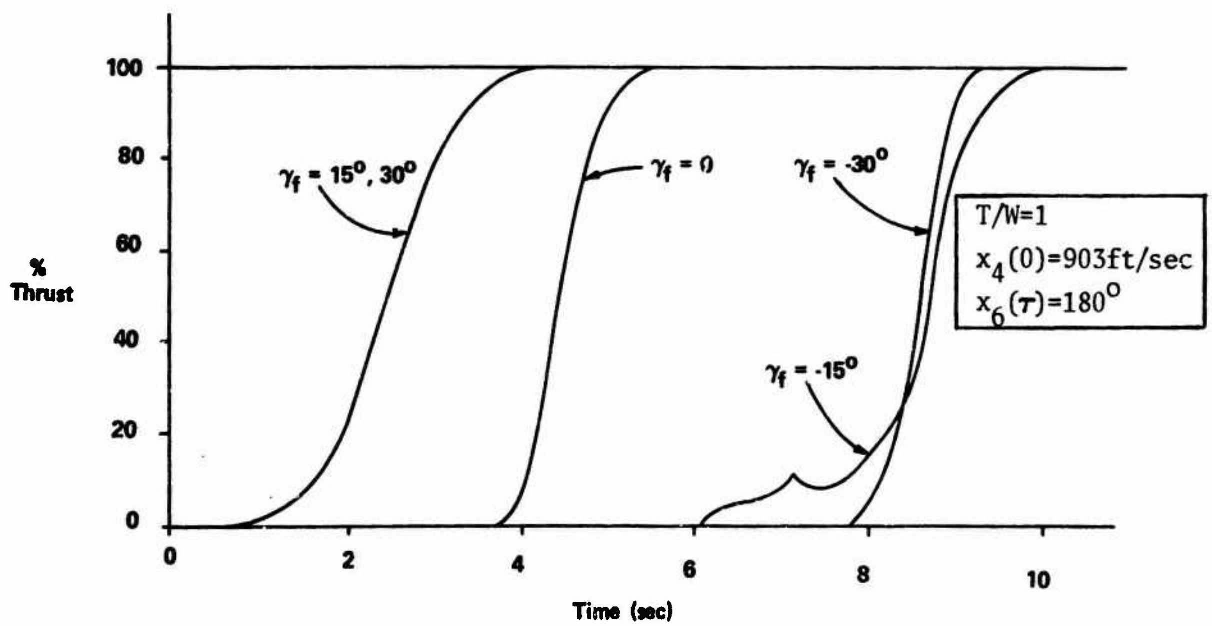

Figure 7. Percent Thrust for Various Final Flight Path Angles

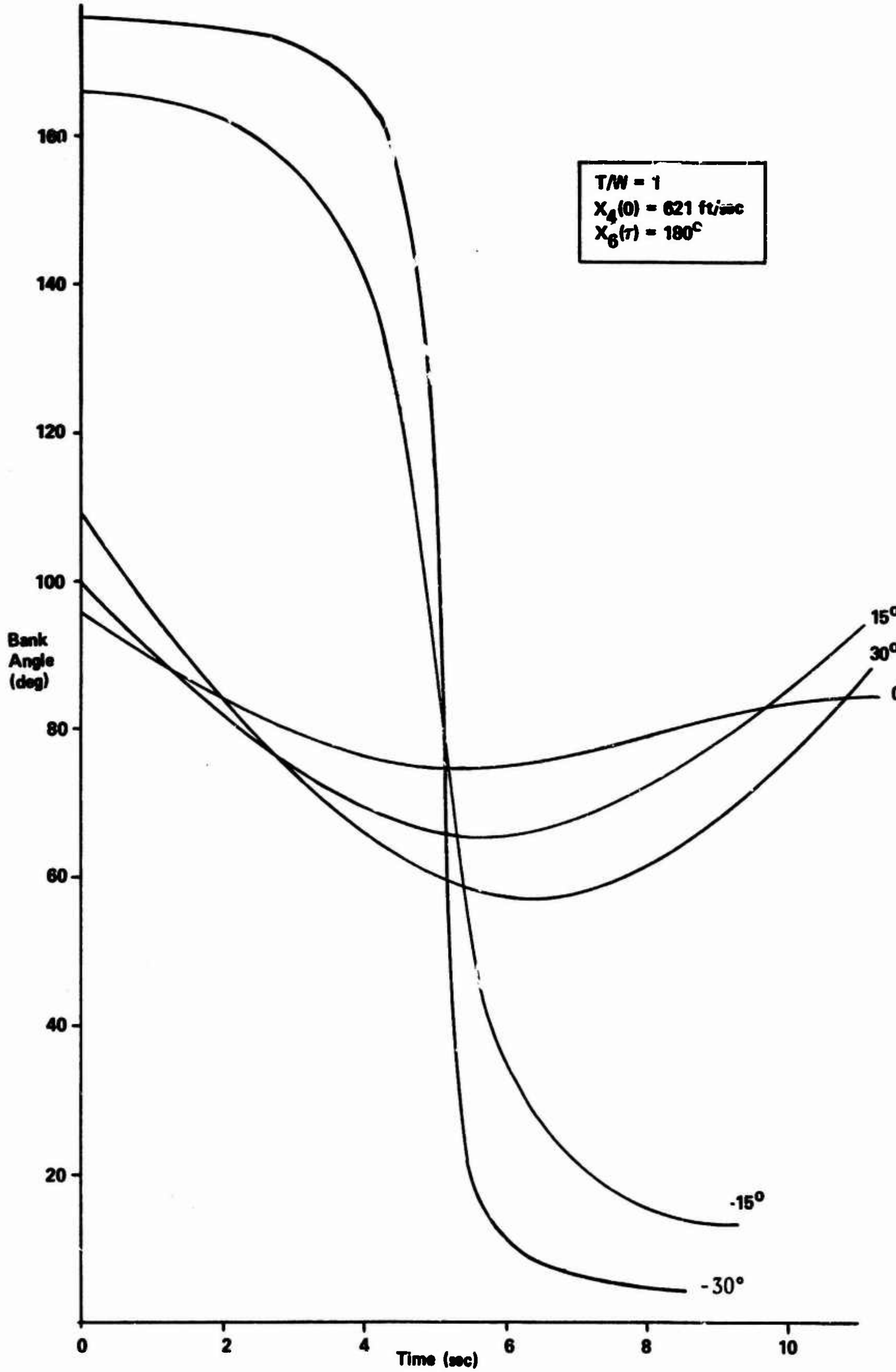

Figure 8. Bank Angle for Various Final Flight Path Angles

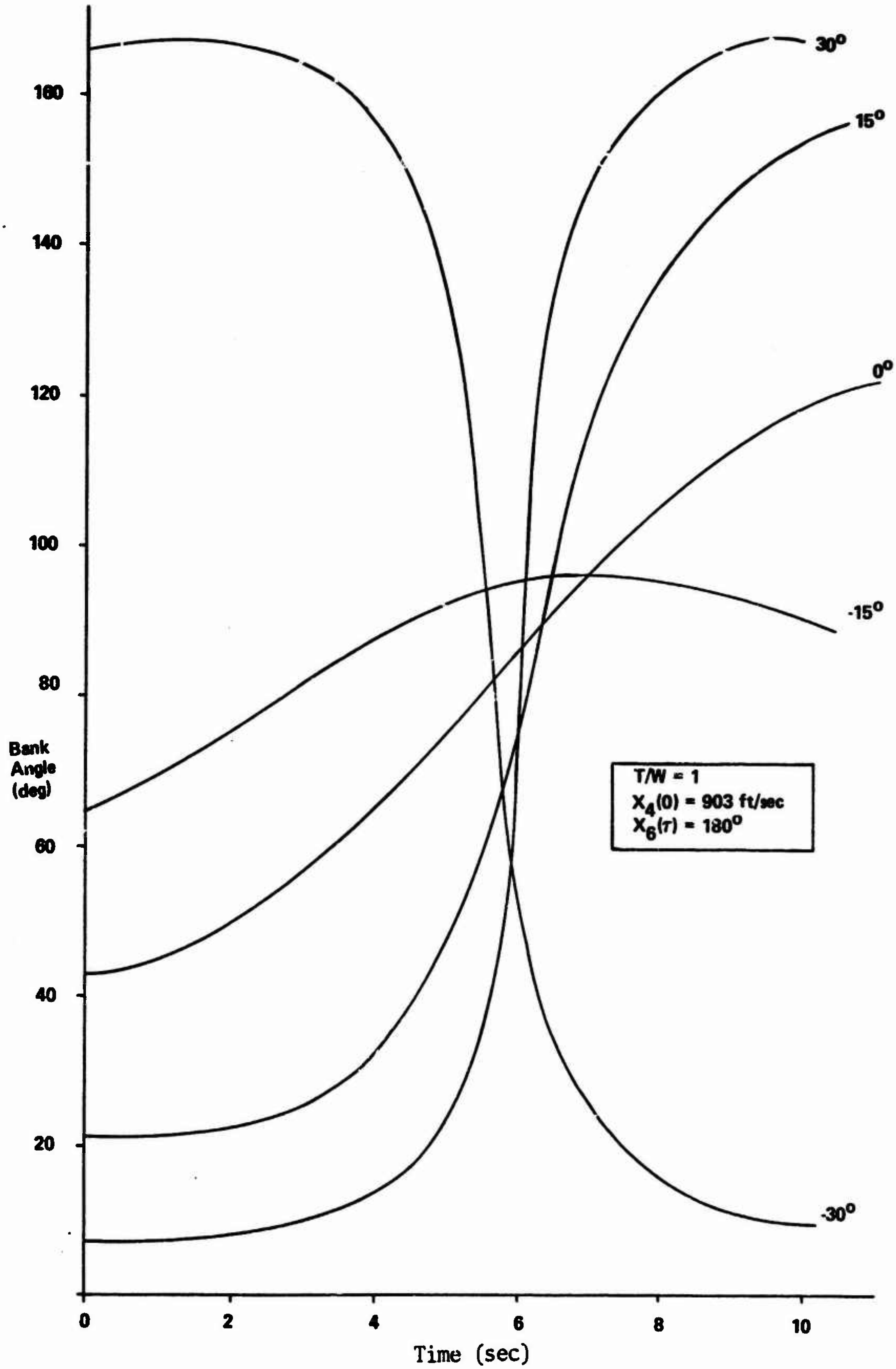

Figure 9. Bank Angle for Various Final Flight Path Angles

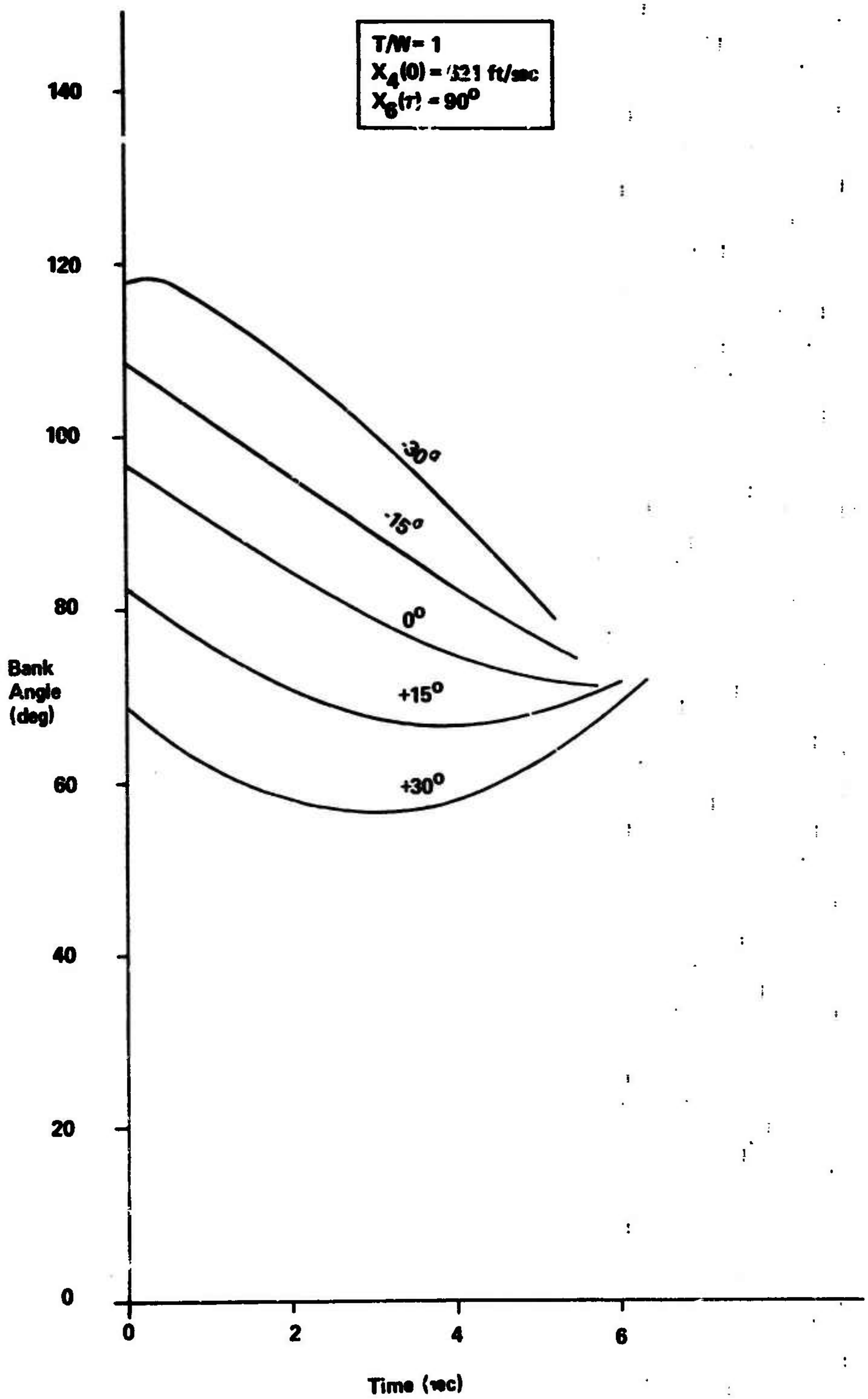

Figure 10. Bank Angle for Various Final Flight Path Angles

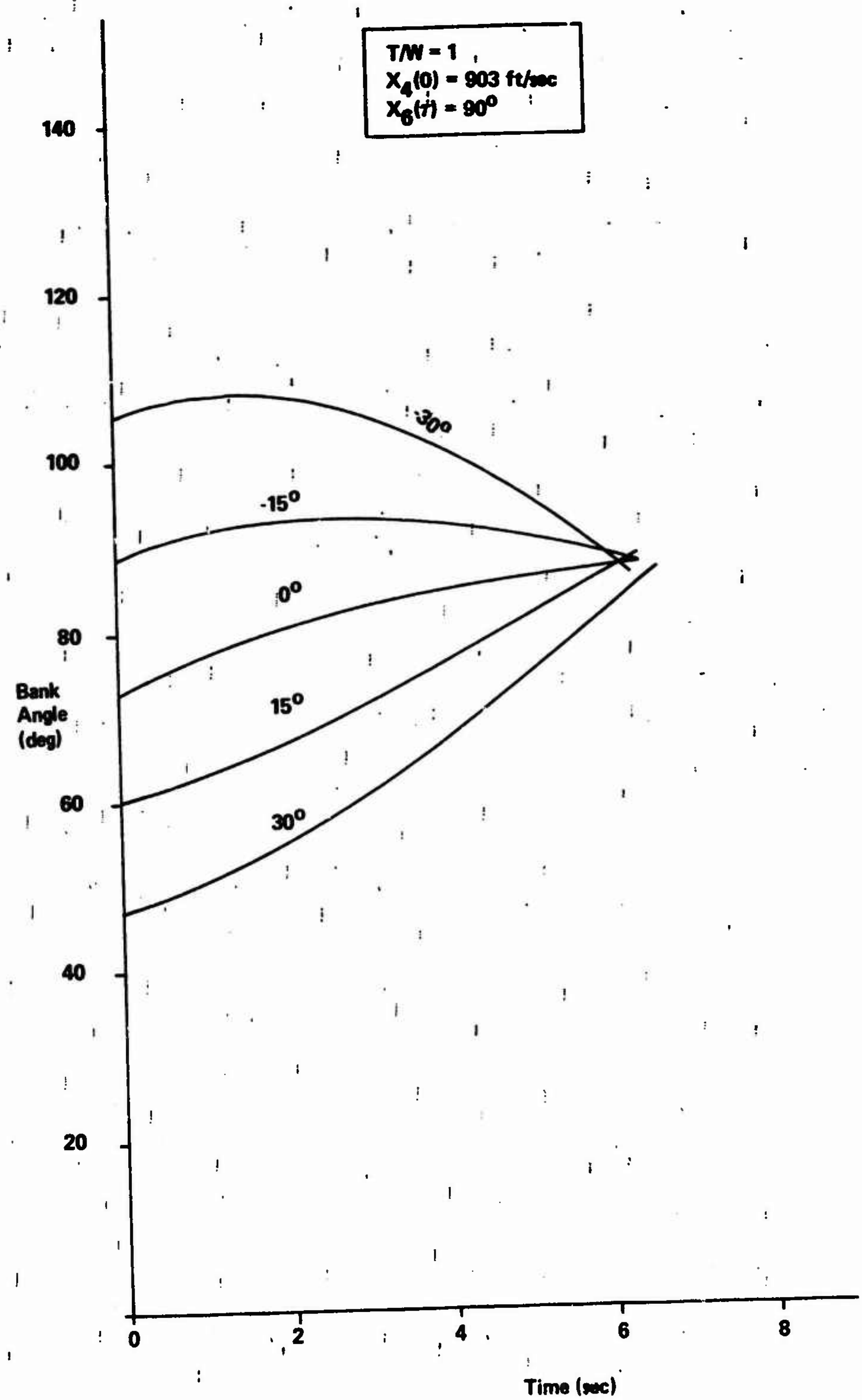

Figure 11. Bank Angle for Various Final Flight Path Angles

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